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Simple and Uniform Method of Obtaining Taylor's, Cayley's, and Lagrange's Series.

BY J. C. GLASHAN, *Ottawa, Canada.*

In a short note in the *American Journal of Mathematics,* Vol. I, p. 287, I have given Taylor's Theorem under the form

$$f(x+a) = \left(1 - \int_0^a da \cdot \frac{d}{dx}\right)^{-1} fx.$$

I find that although this form is excellent as a merely symbolic notation, the operator $\left(1 - \int_0^a da \cdot \frac{d}{dx}\right)^{-1}$ introduces some obscurity in the reasoning if employed in obtaining the theorem. I therefore propose to show the real method adopted in the note and to apply it to obtain Lagrange's Series.

Taylor's Theorem. Let x and a be independent variables, and fy, $f'y$, ... $f^n y$ be continuous from $y = x$ to $y = x + a$.

Since $\quad \frac{d}{da} f(x+a) = \frac{d}{dx} f(x+a) = f'(x+a)$,

$$\therefore \quad f(x+a) = fx + \int_0^a da f'(x+a)$$

$$= fx + \int_0^a da \left\{ f'x + \int_0^a da f''(x+a) \right\}$$

$$= fx + \frac{a}{1} f'x + \int_0^a da \int_0^a da\, f''(x+a)$$

$$= fx + \frac{a}{1} f'x + \frac{a^2}{2} f''x + \left(\int_0^a da\right)^3 f'''(x+a)$$

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$$= fx + \frac{a}{1} f'x + \ldots\ldots + \frac{a^{n-1}}{(n-1)!} f^{n-1}x + \left(\int_0^a da\right)^n f^n(x+a)$$

Cayley's Theorem. Let x, a_0, a_1, a_2, ... be independent variables, and $f^m(y + a_{m+1} + a_{m+2} + \ldots)$ be continuous from $y = x$ to $y = x + a_0 + a_1 + \ldots + a_m$, for all integral values of m from $m = 0$ to $m = n$ inclusive.

Since
$$\frac{d}{da_0} f(x + a_0 + \ldots) = \frac{d}{d(a_0 + a_1)} f(x + a_0 + a_1 + \ldots) = \ldots\ldots$$
$$= \frac{d}{dx} f(x + a_0 + \ldots) = f'(x + a_0 + a_1 + \ldots),$$
$$\therefore f(x + a_0 + a_1 + \ldots) = f(x + a_1 + a_2 + \ldots) + \int_0^{a_0} da_0 f'(x + a_0 + a_1 + \ldots)$$
$$= f(x + a_1 + a_2 + \ldots)$$
$$+ \int_0^{a_0} da_0 \left\{ f'(x + a_2 + \ldots) + \int_0^{a_0 + a_1} d(a_0 + a_1) f''(x + a_0 + a_1 + \ldots) \right\}$$
$$= f(x + a_1 + a_2 + \ldots) + \frac{[a]^1}{1} f'(x + a_2 + \ldots)$$
$$+ \int_0^{a_0} da_0 \int_0^{a_0 + a_1} d(a_0 + a_1) f''(x + a_0 + a_1 + \ldots)$$
$$= f(x + a_1 + a_2 + \ldots) + \frac{[a]^1}{1} f'(x + a_2 + \ldots) + \frac{[a]^2}{2} f''(x + a_3 + \ldots)$$
$$+ \int_0^{a_0} da_0 \int_0^{a_0 + a_1} d(a_0 + a_1) \int_0^{a_0 + a_1 + a_2} d(a_0 + a_1 + a_2) f'''(x + a_0 + a_1 + \ldots)$$

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$$= f(x + a_1 + a_2 + \ldots) + \frac{[a]^1}{1} f'(x + a_2 + \ldots) + \ldots\ldots$$
$$+ \frac{[a]^{n-1}}{(n-1)!} f^{n-1}(x + a_n + \ldots) + \left[\int_0^a da\right]^n f^n(x + a_0 + a_1 + \ldots)$$
in which
$$\left[\int_0^a da\right]^n U \equiv \int_0^{a_0} da_0 \int_0^{a_0 + a_1} d(a_0 + a_1) \ldots \int_0^{a_0 + a_1 + \ldots + a_{n-1}} d(a_0 + a_1 + \ldots + a_{n-1}) U;$$
and $[a]^n \equiv n! \left[\int_0^a da\right]^n =$ the result of retaining only those terms of the expansion of $(a_0 + a_1 + \ldots + a_{n-1})^n$ of the form
$$Ca_0^\alpha (a_1 + a_2 + \ldots + a_\alpha)^\beta (a_{\alpha+1} + a_{\alpha+2} + \ldots + a_{\alpha+\beta})^\gamma \ldots\ldots$$
which $\equiv$ the result of rejecting all terms of the same expansion, of the form
$$Ca_0^{n_0} a_1^{n_1} a_2^{n_2} \ldots\ldots$$
in which $n_{n-1} > 1$, $n_{n-2} + n_{n-1} > 2$, ... $n_1 + n_2 + \ldots + n_{n-1} > n - 1$.

Writing x for $x + a_0 + a_1 + \ldots\ldots$, the theorem becomes

$$fx = f(x - a_0) + \frac{[a]^1}{1} f'(x - a_0 - a_1) + \ldots\ldots$$
$$+ \frac{[a]^{n-1}}{(n-1)!} f^{n-1}(x - a_0 - a_1 - \ldots - a_{n-1}) + R.$$

Let $\{a\}^n \equiv$ the result of retaining only those terms of the expansion of $(a_0 + a_1 + a_2 + \ldots\ldots + a_{n-1})^n$ of the form

$$C a_0^\alpha a_\alpha^\beta a_{\alpha+\beta}^\gamma \ldots\ldots$$

and affecting each term with the sign $(-1)^{n-m}$ where $m = \frac{\alpha}{\alpha} + \frac{\beta}{\beta} + \frac{\gamma}{\gamma} + \ldots$, and the theorem may be written

$$fx = f(x - a_0) + \frac{\{a\}^1}{1} f'(x - a_1) + \ldots\ldots + \frac{\{a\}^{n-1}}{(n-1)!} f^{n-1}(x - a_{n-1}) + R.$$

At the time of the publication of the above theorem in my note entitled *An Extension of Taylor's Theorem*, (Vol. I, p. 287,) I believed it to be new, but the following note published in *Quart. Journ. Math.*, XIV., 53 (two years before mine), shows the theorem had been given long before by Professor Cayley.

"I wish to put on record the following theorem, given by me as a Senate-House Problem, January, 1851.

If $\{\alpha + \beta + \gamma \ldots\}^p$ denote the expansion of $(\alpha + \beta + \gamma \ldots)^p$, retaining those terms $N\alpha^a\beta^b\gamma^c \ldots$ only in which

$$b + c + d \ldots \not> p - 1, \quad c + d \ldots \not> p - 2, \quad \&c., \&c.,$$

then

$$x^n = (x + \alpha)^n - n\{\alpha\}^1(x + \alpha + \beta)^{n-1} + \frac{1}{2}n(n-1)\{\alpha + \beta\}^2(x + \alpha + \beta + \gamma)^{n-2}$$
$$- \frac{1}{6}n(n-1)(n-2)\{\alpha + \beta + \gamma\}^3(x + \alpha + \beta + \gamma + \delta)^{n-3} + \&c.$$

The theorem, in a somewhat different and imperfectly stated form, is given, Burg, *Crelle*, t. I. (1826), p. 368, as a generalization of Abel's theorem

$$(x + \alpha)^n = x^n + n\alpha(x + \beta)^{n-1} + \frac{1}{2}n(n-1)\alpha(\alpha - 2\beta)(x + 2\beta)^{n-2}$$
$$+ \frac{1}{6}n(n-1)(n-2)\alpha(\alpha - 3\beta)^2(x + 3\beta)^{n-3} + \&c."$$

Professor Cayley's proof of this theorem, which we have taken the liberty to call by his name, is given in *Solutions of the Cambridge Senate-House Problems*, 1848–1851, pp. 94–96.

Lagrange's Theorem. Let x and a be independent variables, $u = x + a\phi u$, and fu, $f'u$, ... $f^n u$ continuous from $u = x$ to $u = x + a\phi u$.

Then
$$\frac{dfu}{da} = \phi u \frac{dfu}{dx} = \phi u f'u \frac{du}{dx},$$
$$^{a=0}\left(\frac{d}{da}\right)^n fu = \left(\frac{d}{dx}\right)^{n-1}\{(\phi x)^n f'x\},$$

whence
$$fu = fx + \int_0^a da\left(\frac{dfu}{da}\right)$$
$$= fx + \int_0^a da\left\{\phi u f'u \frac{du}{dx}\right\}$$
$$= fx + \int_0^a da\left[\phi x f'x + \int_0^a da\left\{\frac{d}{da}\left(\phi u f'u \frac{du}{dx}\right)\right\}\right]$$
$$= fx + \frac{a}{1}\phi x f'x + \left(\int_0^a da\right)^2 \frac{d}{dx}\left\{(\phi u)^2 f'u \frac{du}{dx}\right\}$$
$$\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots$$
$$= fx + \frac{a}{1}\phi x f'x + \ldots\ldots + \frac{a^{n-1}}{(n-1)!}\left(\frac{d}{dx}\right)^{n-2}\left\{(\phi x)^{n-1} f'x\right\}$$
$$+ \left(\int_0^a da\right)^n \left[\left(\frac{d}{dx}\right)^{n-1}\left\{(\phi u)^n f'u \frac{du}{dx}\right\}\right].$$

It may be noticed in this connection that Lagrange's Series can, like Taylor's Theorem, be obtained by integration by parts, and that the remainder can be obtained in the form of a definite integral, thus:—

Let x, a and α be independent variables, and
$$u = x + a\phi u, \qquad v = x + (a - \alpha)\phi v.$$

Then
$$^{\alpha=a}Fv = Fx, \qquad ^{\alpha=0}Fv = Fu,$$
$$\frac{dFv}{da} = -\frac{dFv}{d\alpha}, \text{ and } \frac{d}{d\alpha}\left\{(\phi v)^n \frac{dfv}{dx}\right\} = -\frac{d}{dx}\left\{(\phi v)^{n+1}\frac{dfv}{dx}\right\}.$$

Therefore
$$fx - fu = \int_0^a d\alpha\left(\frac{d}{d\alpha} fv\right)$$
$$fu = fx + \int_0^a d\alpha\left(\phi v \cdot \frac{dfv}{dx}\right)$$

Integrating by parts,

$$\int_0^a d\alpha\left(\phi v\cdot\frac{dfv}{dx}\right)=a\phi x\,.\,f'x-\int_0^a d\alpha\left\{\alpha\frac{d}{d\alpha}\left(\phi v\cdot\frac{dfv}{dx}\right)\right\}$$
$$=a\phi x\,.\,f'x+\frac{1}{2}\int_0^a d\alpha\left[\frac{d(\alpha^2)}{d\alpha}\cdot\frac{d}{dx}\left\{(\phi v)^2\frac{dfv}{dx}\right\}\right].$$

Similarly

$$\frac{1}{2}\int_0^a d\alpha\left[\frac{d(\alpha^2)}{d\alpha}\cdot\frac{d}{dx}\left\{(\phi v)^2\frac{dfv}{dx}\right\}\right]=\frac{a^2}{2}\frac{d}{dx}\left\{(\phi x)^2 f'x\right\}$$
$$+\frac{1}{2\,.\,3}\int_0^a d\alpha\left[\frac{d(\alpha^3)}{d\alpha}\left(\frac{d}{dx}\right)^2\left\{(\phi v)^3\frac{dfv}{dx}\right\}\right];$$

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$$\frac{1}{n!}\int_0^a d\alpha\left[\frac{d(\alpha^n)}{d\alpha}\left(\frac{d}{dx}\right)^{n-1}\left\{(\phi v)^n\frac{dfv}{dx}\right\}\right]=$$
$$\frac{a^n}{n!}\left(\frac{d}{dx}\right)^{n-1}\left\{(\phi x)^n f'x\right\}+\frac{1}{n!}\int_0^a d\alpha\left[\alpha^n\left(\frac{d}{dx}\right)^n\left\{(\phi v)^{n+1}\frac{dfv}{dx}\right\}\right].$$

$$\therefore\quad fu=fx+\frac{a}{1}\phi x\,.\,f'x+\frac{a}{2}\frac{d}{dx}\left\{(\phi x)^2 f'x\right\}+\,.\,.\,.\,.\,.\,.\,.\,.$$
$$+\frac{a^n}{n!}\left(\frac{d}{dx}\right)^{n-1}\left\{(\phi x)^n f'x\right\}+\frac{1}{n!}\int_0^a d\alpha\left[\alpha^n\left(\frac{d}{dx}\right)^n\left\{(\phi v)^{n+1}\frac{dfv}{dx}\right\}\right].$$

This form of the remainder corresponds to that of Taylor's Theorem obtained by integration by parts. (See also xx_1 of the article following this, entitled *Forms of Rolle's Theorem.*) This form can easily be reduced to that obtained by Zolotareff's method, (*Williamson's Integral Calculus*, 3d edition, p. 159); thus:

$$\int_0^a d\alpha\left[\alpha^n\left(\frac{d}{dx}\right)^n\left\{(\phi v)^{n+1}\frac{dfv}{dx}\right\}\right]=\left(\frac{d}{dx}\right)^n\int_0^a d\alpha\left\{(\alpha\,.\,\phi v)^n\,\phi v f'v\,\frac{dv}{dx}\right\}$$
$$=\left(\frac{d}{dx}\right)^n\int_a^0 d\alpha\left\{(a\,.\,\phi v+x-v)^n f'v\,\frac{dv}{d\alpha}\right\}$$
$$=\left(\frac{d}{dx}\right)^n\int_x^u dv\left\{(a\,.\,\phi v+x-v)^n f'v\right\}.$$

P. S.—I find that a form for the remainder in Lagrange's Series was given by Schlömilch, *Liouville*, III_2, 390, (1858.)